On topological and geometric (19_4) configurations

Jürgen Bokowski, Vincent Pilaud

ON TOPOLOGICAL AND GEOMETRIC (19_4) CONFIGURATIONS

JÜRGEN BOKOWSKI AND VINCENT PILAUD[‡]

Abstract. An (n_k) configuration is a set of n points and n lines such that each point lies on k lines while each line contains k points. The configuration is geometric, topological, or combinatorial depending on whether lines are considered to be straight lines, pseudolines, or just combinatorial lines. The existence and enumeration of (n_k) configurations for a given k has been subject to active research. A current front of research concerns geometric (n_4) configurations: it is now known that geometric (n_4) configurations exist for all $n \geq 18$, apart from sporadic exceptional cases. In this paper, we settle by computational techniques the first open case of (19_4) configurations: we obtain all topological (19_4) configurations among which none are geometrically realizable.

1. Introduction

An *(n_k) configuration* is formed by a set P of n *points* and a set L of n *lines* such that each point of P lies on precisely k lines of L while each line of L contains precisely k points of P. The different possible meanings for points and lines define different notions of configurations:

(i) For *geometric configurations*, points and lines are ordinary points and lines in the real projective plane $\mathbb{P}$.
(ii) For *topological configurations*, points are ordinary points in $\mathbb{P}$, but lines are *pseudolines*, *i.e.* non-separating simple closed curves of $\mathbb{P}$ which cross pairwise precisely once.
(iii) For *combinatorial configurations*, we just consider abstract points and lines, together with an incidence relation such that no two distinct points are incident to two distinct lines. Equivalently, combinatorial (n_k) configurations can be described as k-regular bipartite graphs on $2n$ vertices with girth at least 6.

Famous examples of such configurations are represented in Figure 1. These examples reflect the long history of (n_k) configurations, and their connections to projective incidence theorems and realizability problems. A detailed survey on configurations including historical perspectives and careful references to the literature can be found in the recent monograph of B. Grünbaum [Grü09].

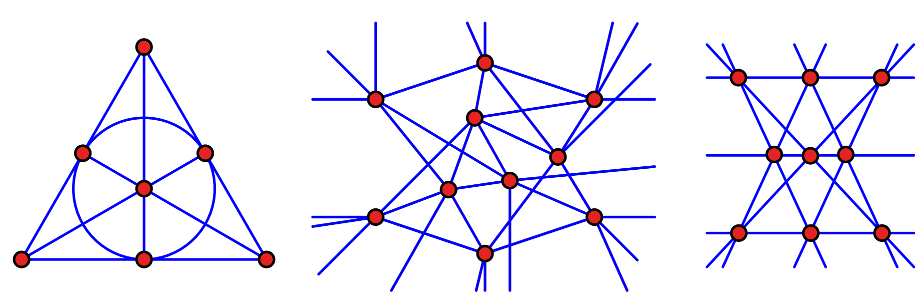

Figure 1. Fano's combinatorial (7_3) configuration (left), Kantor's topological (10_3) configuration (center), and Pappus' geometric (9_3) configuration (right).

The first question on configurations raised in B. Grünbaum's monograph is to describe, for a fixed integer k, for which values of n do combinatorial, topological, and geometric (n_k) configurations exist. This question is completely settled for $k \leq 3$, extensively studied for $k = 4$, and still

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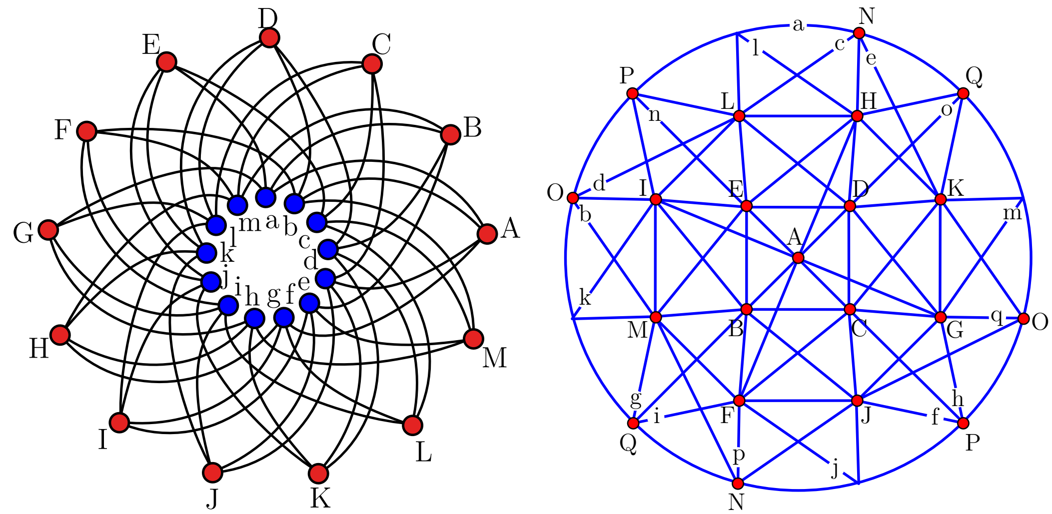

FIGURE 2. The incidence graph (Levi graph) of the first combinatorial (13_4) configuration (left) and the first topological (17_4) configuration [BS13] (right).

widely open for $k \geq 5$. When $k = 4$, the existence of geometric (n_4) configurations has been proved for all sufficiently large integers n by various geometric constructions, most of them using non-trivial symmetry groups. These constructions are surveyed in [Grü09, Chapter 3]. The remaining integers n have been treated individually, with ad-hoc constructions or arguments to prove or disprove the existence of (n_4) configurations. The current state of knowledge is the following: combinatorial (n_4) configurations exist iff $n \geq 13$, topological (n_4) configurations exist iff $n \geq 17$ [BS05, BGS09] and geometric (n_4) configurations exist iff $n \geq 18$ [Grü00, Grü02, Grü06, BS13], with the possible exceptions[1] of $n = 19$, 22, 23, 26, 37 and 43. To illustrate these results, we have represented in Figures 2 and 3 the first examples of (n_4) configurations: Figure 2 shows the incidence relation of the first combinatorial (13_4) configuration and the first topological (17_4) configuration (antipodal points of the circle are identified), while Figure 3 shows the only two geometric (18_4) configurations [BS05, BP13] (some points are at infinity to obtain more symmetric pictures).

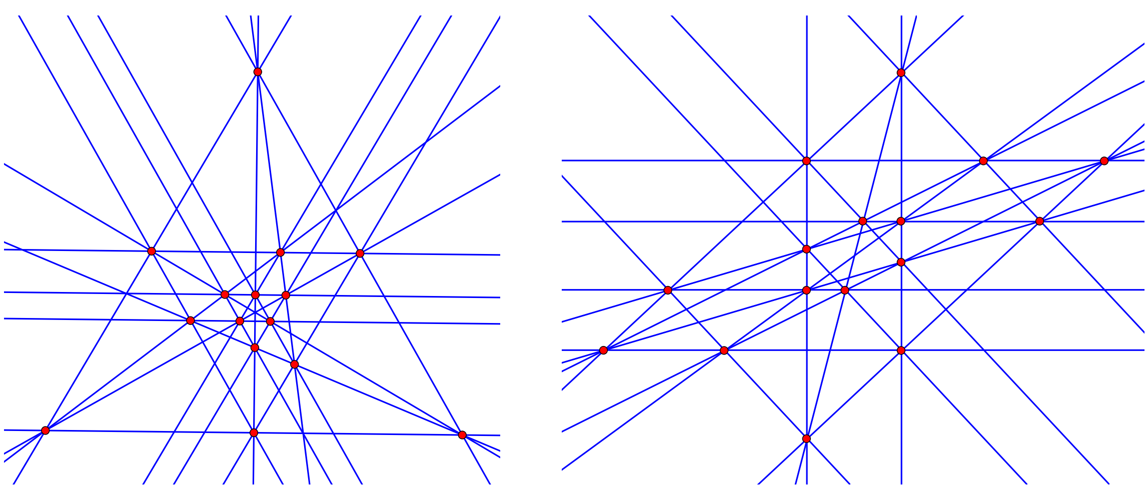

FIGURE 3. The two geometric (18_4) configurations [BS05, BP13].

[1]In fact, there are currently only 3 remaining cases. Indeed, case $n = 19$ is treated in detail in this paper, while cases $n = 37$ and 43, as well as several by-products of our investigation on small (n_k) configurations, will be discussed in a separate paper, to keep the present paper short and focused on topological and geometric (19_4) configurations.

This paper settles the case of (19_4) configurations. All 269 224 653 combinatorial (19_4) configurations were recently enumerated in [OC12]. It should however be clear that searching for topologically or geometrically realizable configurations in this list would be like looking for a needle in a haystack. We have instead developed and implemented in [BP13] an algorithm to generate directly all topological (19_4) configurations up to combinatorial equivalence, which does not start from the list of all combinatorial (19_4) configurations. Using this algorithm as a black box, we report in Section 2 on the list of all 4028 topological (19_4) configurations, with a particular attention to their isomorphism group. We then present in Section 3 the so-called construction sequence method which we use on the one hand to search for subconfigurations in a configuration and on the other hand to test the geometric realizability of a configuration. Using this method, we surprisingly conclude that there is no geometric (19_4) configuration.

2. Topological (19_4) configurations

We developed in [BP13] an algorithm to enumerate directly all topological (n_k) configurations without enumerating first the combinatorial (n_k) configurations. This algorithm, implemented in JAVA, enumerates all topological (19_4) configurations in approximately two weeks[2]:

Result 1 ([BP13])**.** *There are precisely* 4 028 *topological* (19_4) *configurations up to combinatorial equivalence. Among them,* 222 *are self-dual.*

Studying this list, we can already answer B. Grünbaum's problem to find symmetric topological (19_4) configurations [Grü09, p. 169, Question 5]. First, the combinatorial automorphism groups of these configurations are distributed as follows:

group G	1	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	D_8	Total
number of (19_4) configurations with automorphism group G	3 726	283	14	2	3	4028

By *combinatorial automorphisms* of a configuration (P, L) we mean the automorphisms of its Levi graph. Remember that the *Levi graph* of a point – line configuration (P, L) is the bipartite graph whose nodes are the elements of $P \sqcup L$ and whose edges relate incident elements. An example is given in Figure 2 (left). An automorphism of the Levi graph of a configuration (P, L) is a *preserving automorphism* of the configuration when it fixes the two maximal independent sets P and L, and a *self-duality* of the configuration if it exchanges P and L. For example, in Figure 2 (left), the rotation of angle $2\pi/13$ is a preserving automorphism, while the transformation (A,a)(B,b) . . . (L,l)(M,m) is a self-duality.

We then tried to realize the five configurations whose combinatorial automorphism group has order 8 in such a way that their preserving automorphism are realized as isometries, and that the self-dualities are self-polarities of the configuration. A *self-polarity* of a topological configuration (P, L) is a self-duality $^* : (P, L) \mapsto (L^*, P^*)$ which respects cyclic orders: if the points $p_1, \dots, p_k \in P$ appear in cyclic order along a pseudoline $\ell \in L$, then the dual pseudolines $p_1^*, \dots, p_k^* \in P^*$ must appear in cyclic order around the dual point $\ell^* \in L^*$, and similarly if the lines $\ell_1, \dots, \ell_k \in L$ appear in cyclic order around a point $p \in P$, then the dual points $\ell_1^*, \dots, \ell_k^* \in L^*$ must appear in cyclic order along the dual line $p^* \in P^*$. For example, the transformation (A,a)(B,b) . . . (R,r)(S,s) is a self-polarity of the topological configuration of Figure 2 (right).

The most symmetric topological (19_4) configuration that we obtained is represented in Figure 4 and 7. Its point – line incidences are given by the following table:

lines	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s
	B	A	A	A	A	H	I	B	C	D	E	F	P	E	D	C	B	P	Q
points	C	H	I	J	K	L	L	F	G	I	H	G	Q	Q	P	R	S	M	M
in lines	E	D	E	B	C	O	N	L	L	K	J	I	S	O	N	M	M	K	J
	D	Q	P	O	N	S	R	K	J	S	R	H	R	G	F	O	N	G	F

[2]Computation time on a 2.4 GHz Intel Core 2 Duo processor with 4Go of RAM.

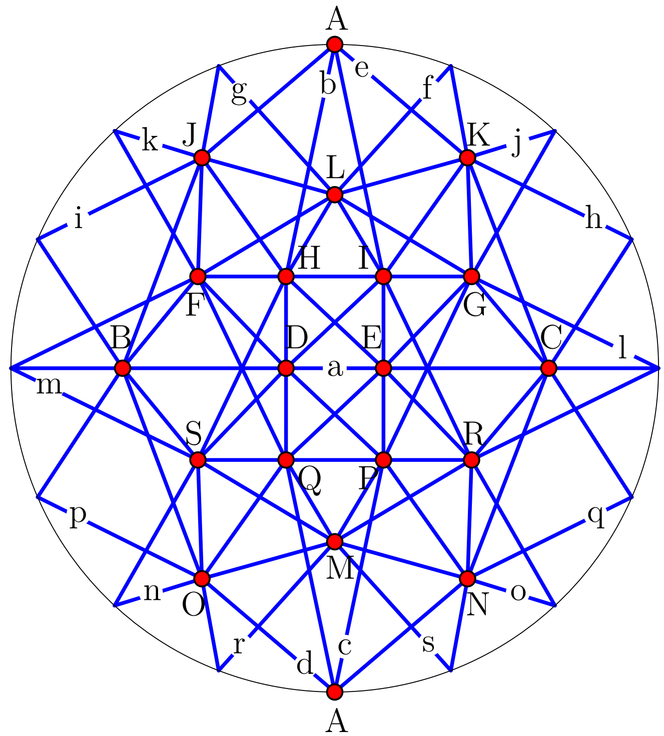

FIGURE 4. A topological (19_4) configuration whose isometry group is $\mathbb{Z}_2 \times \mathbb{Z}_2$, and with an additional self-polarity.

We have labeled the points and pseudolines of this configuration in such a way that:

- the action on the points of the vertical and horizontal reflexions of the picture are respectively given by the permutations
  (A)(B,C)(D,E)(F,G)(H,I)(J,K)(L)(M)(N,O)(P,Q)(R,S) and
  (A)(B)(C)(D)(E)(F,S)(G,R)(H,Q)(I,P)(J,O)(K,N)(L,M);
- the permutation (A,a)(B,b) ... (R,r)(S,s) is a self-polarity.

The combinatorial automorphism group of the configuration is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, and is thus completely realized in the picture: it is the direct product between the rectangle isometry group $\mathbb{Z}_2 \times \mathbb{Z}_2$ and the self-polarity group $\mathbb{Z}_2$. This example answers positively B. Grünbaum's problem [Grü09, p. 169, Question 5]:

Result 2. *There exist topological* (19_4) *configurations realized with non-trivial isomorphism groups.*

Besides the configuration of Figure 4, there are four other topological (19_4) configurations with combinatorial automorphism group of order 8. They are represented in Figures 5, 6, and 7. We have labeled the points and lines of these configurations such that the non-trivial isometry is given by a nice permutation and the self-duality by the permutation (A,a)(B,b) ... (R,r)(S,s). We detail below for each configuration its point – line incidences, and a generating system of its combinatorial automorphism group.

Observe that for these four configurations, we did not manage to obtain a representation where the full combinatorial automorphism group acts as isometries or polarities. It is not surprising for the configurations whose automorphism group is the dihedral group D_8 since it is already impossible to construct a topological (19_4) configuration with a C_4 symmetry. Indeed, since 19 is odd, an automorphism of order 4 should fix a line ℓ and rotate the four segments of ℓ delimited by the points of the configuration. This is impossible since these four segments cross 6 lines of the configuration, and 6 is not a multiple of 4.

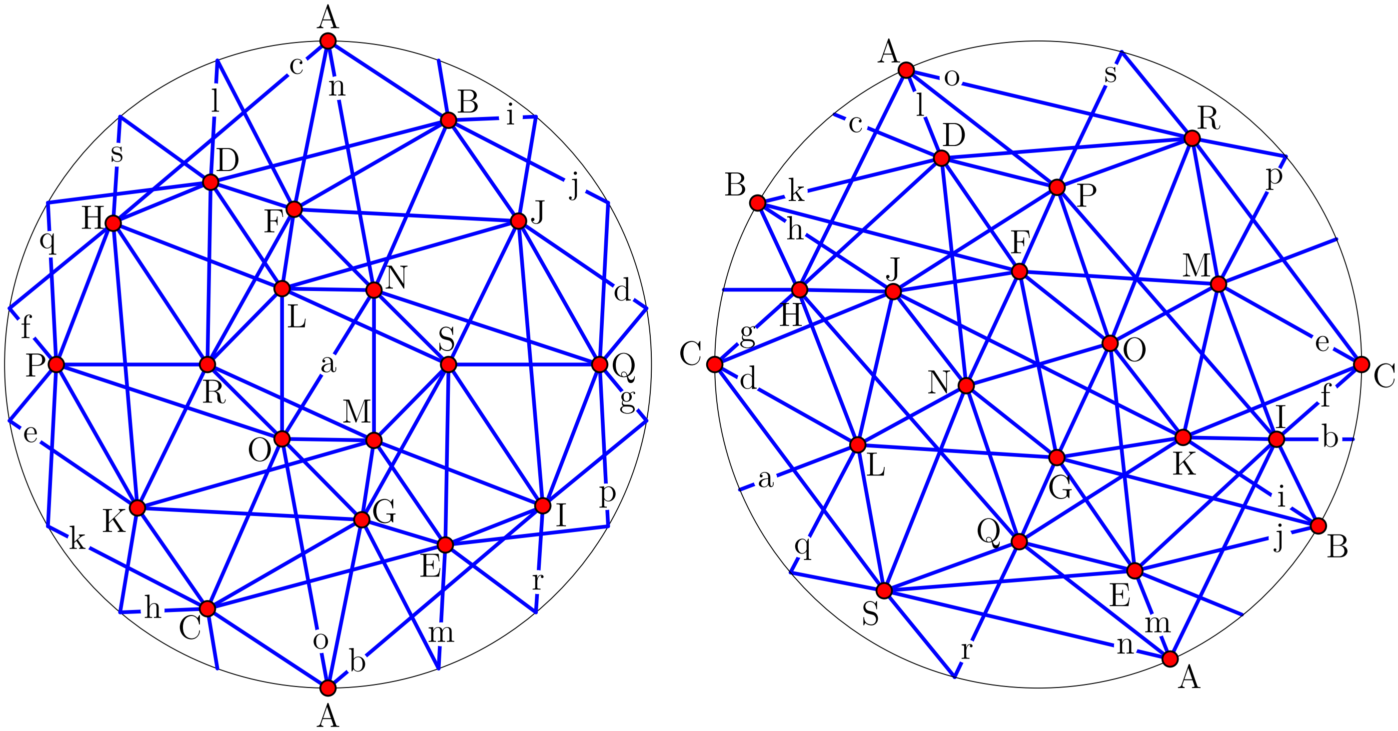

FIGURE 5. Two centrally symmetric topological (19_4) configurations.

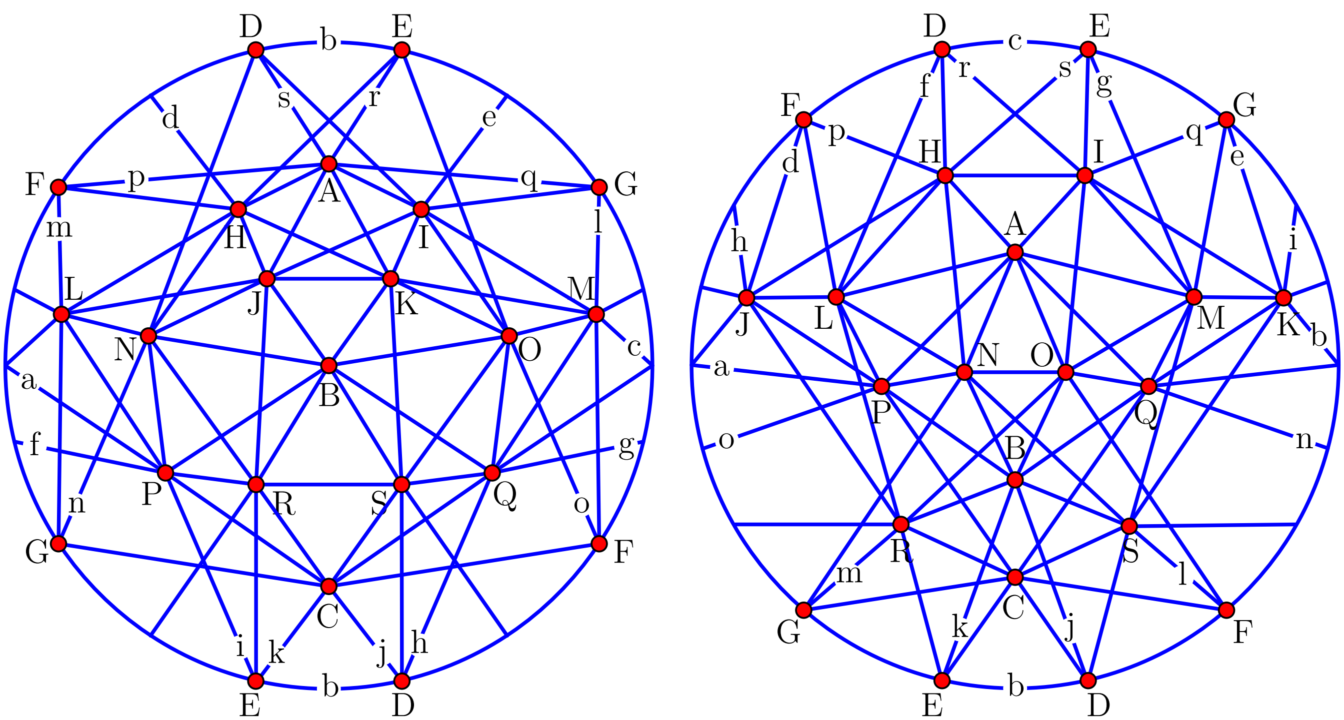

FIGURE 6. Two topological (19_4) configurations with vertical symmetry.

(19_4) configuration in Figure 5 (left)

lines	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s
	B	A	A	J	K	I	H	C	B	B	C	D	E	A	A	D	E	D	E
points	C	B	C	L	M	M	L	G	F	D	E	G	F	G	F	F	G	I	H
in lines	O	J	K	R	S	O	N	S	R	H	I	O	N	M	L	J	K	S	R
	N	I	H	P	Q	P	Q	J	K	P	Q	R	S	N	O	Q	P	L	M

Automorphism group generated by:

(A)(B,C)(D,E)(F,G)(H,I)(J,K)(L,M)(N,O)(P,Q)(R,S)
(A)(B,O)(C,N)(D)(E)(F,I)(G,H)(J,L)(K,M)(P,R)(Q,S)

together with the self-duality (A,a)(B,b) . . . (R,r)(S,s), which is not a self-polarity.

(19_4) configuration in Figure 5 (right)

lines	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s
	L	H	D	C	C	C	C	B	B	B	B	A	A	A	A	K	J	G	F
points	M	I	E	K	J	I	H	G	F	E	D	D	E	H	I	M	L	O	N
in lines	O	K	G	G	F	E	D	N	O	Q	P	N	O	L	M	S	R	R	S
	N	J	F	L	M	S	R	J	K	H	I	Q	P	S	R	Q	P	Q	P

Automorphism group generated by:

(A)(B)(C)(D,E)(F,G)(H,I)(J,K)(L,M)(N,O)(P,Q)(R,S)
(A)(B,C)(D,I,E,H)(F,K,G,J)(L,N,M,O)(P,S,Q,R)

together with the self-polarity (A,a)(B,b) . . . (R,r)(S,s).

(19_4) configuration in Figure 6 (left)

lines	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s
	P	D	J	B	B	B	B	D	E	C	C	C	C	G	F	A	A	A	A
points	Q	E	K	J	K	O	N	I	H	D	E	G	F	I	H	F	G	E	D
in lines	S	G	M	H	I	M	L	O	N	N	O	M	L	J	K	M	L	R	S
	R	F	L	S	R	P	Q	Q	P	R	S	Q	P	N	O	I	H	J	K

Automorphism group generated by:

(A)(B)(C)(D,E)(F,G)(H,I)(J,K)(L,M)(N,O)(P,Q)(R,S)
(A,C)(B)(D,F,E,G)(H,N,I,O)(J,Q,K,P)(L,R,M,S)

together with the self-duality (A,a)(B,b) . . . (R,r)(S,s), which is not a self-polarity.

(19_4) configuration in Figure 6 (right)

lines	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s
	N	H	D	C	C	C	C	B	B	B	B	F	G	A	A	A	A	D	E
points	O	I	E	F	G	D	E	P	Q	D	E	L	M	L	M	H	I	I	H
in lines	Q	K	G	J	K	L	M	J	K	H	I	N	O	J	K	F	G	M	L
	P	J	F	R	S	P	Q	S	R	N	O	S	R	Q	P	O	N	S	R

Automorphism group generated by:

(A)(B)(C)(D,E)(F,G)(H,I)(J,K)(L,M)(N,O)(P,Q)(R,S)
(A,B)(C)(D,F,E,G)(H,O,I,N)(J,Q,K,P)(L,R,M,S)

together with the self-polarity (A,a)(B,b) . . . (R,r)(S,s).

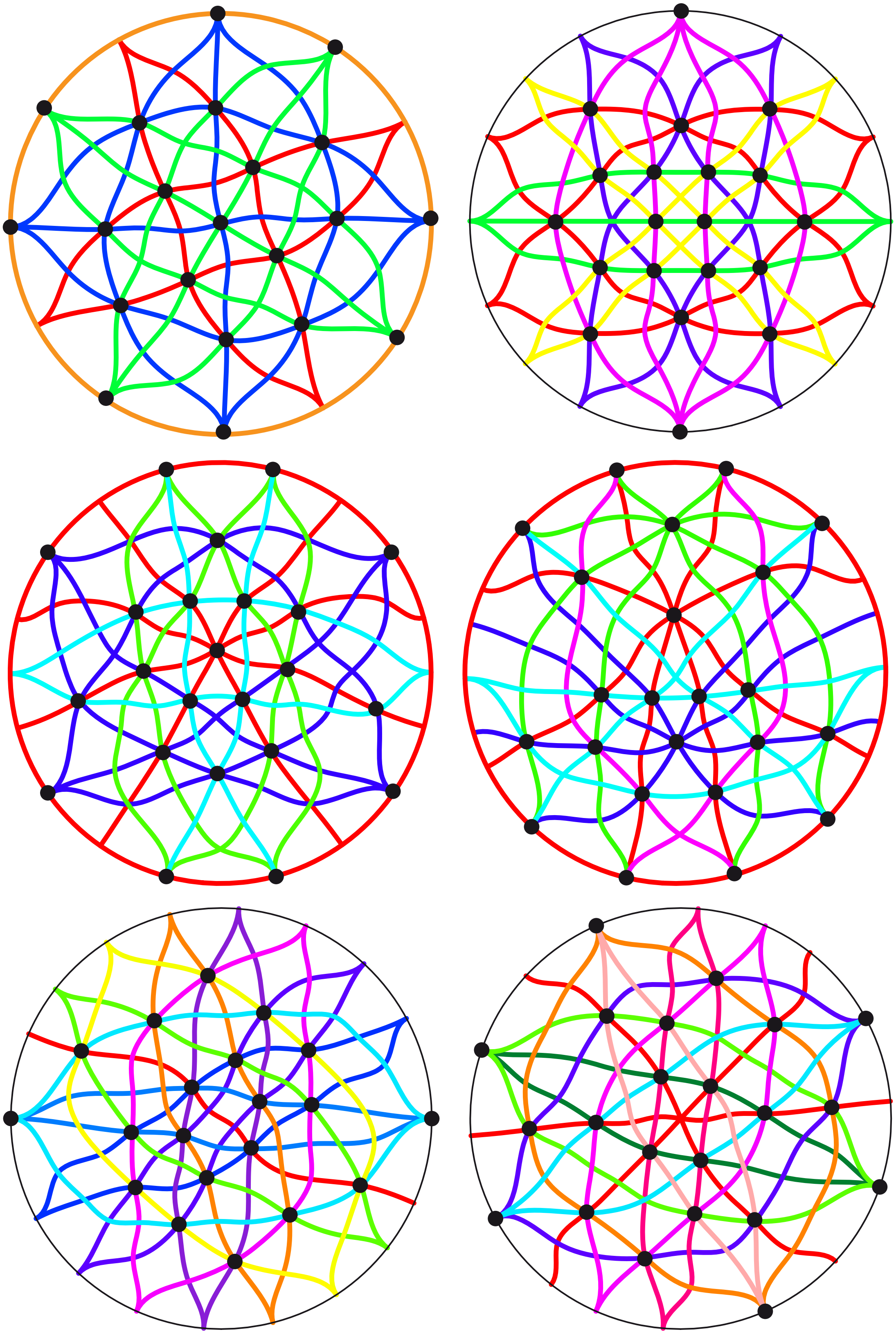

FIGURE 7. Smooth pictures for the topological (17_4) and (19_4) configurations of Figures 2 (right), 4, 5, and 6, obtained by omawin (research software available upon request to the first author).

3. Geometric (19_4) configurations

In this section, we present our techniques to search for geometric realizations of the topological (19_4) configurations discussed in Section 2. Observe already that we can restrict our attention to 2 125 topological (19_4) configurations, keeping only one representative in each duality class. Our main tool is the *construction sequence method*, which enables us to search for subconfigurations in a configuration and to test geometric realizability of configurations. We present this method in detail below although it is a classical folklore when programing on the projective plane (it is used *e.g.* in most dynamic geometric softwares such as CINDERELLA or THE GEOMETER'S SKETCHPAD).

Construction sequences — Consider the problem of searching an incidence-preserving embedding ϕ of a small finite point – line configuration (P, L) into a large point – line configuration (Π, Λ). Note that this problem covers two relevant situations that we will detail later on:

(i) if (Π, Λ) is a finite point – line configuration, then we are searching for subconfigurations isomorphic to (P, L) in a configuration (Π, Λ).
(ii) if Π is the set of all points and Λ the set of all lines of the plane, then we are testing the geometric realizability of (P, L).

In both situations, it is natural to start fixing the image under ϕ of an arbitrary projective base P_1 of (P, L), then construct the image of the set L_1 of all lines of L joining two points of P_1, then the image of the set P_2 of all points of $P \smallsetminus P_1$ contained in at least two lines of L_1, etc. If this procedure finishes, we can easily test whether the final embedding ϕ indeed respects all incidences of (P, L). It might however happen that the procedure described above does not finish: at some point, it might happen that none of the remaining point (or line) is incident with two constructed lines (or points). In this case, we have to consider all possible positions for constructing a new point (or line) before getting back to the procedure.

We formalize this intuitive description as follows. We denote by $p \vee p'$ the unique line of L passing through two points p and p' of P (if it exists). Similarly, let $\ell \wedge \ell'$ be the unique point of P contained in two lines ℓ and ℓ' of L (if it exists). A *projective base* of the combinatorial point – line configuration (P, L) is a set B of four points of P such that for every triple $T \subset B$, there exists a line of L containing precisely two points of T. This ensures that no three points of B can be aligned, even in a larger point – line configuration containing (P, L). A *construction sequence* for (P, L) is a sequence (X_i) of subsets of $P \sqcup L$ such that

(i) X_0 is a projective base of (P, L).
(ii) X_{i+1} is the set of all elements of $P \sqcup L$ not in $\bigcup_{j \le i} X_j$ incident to at least two elements of $\bigcup_{j \le i} X_j$ if it is non-empty. Otherwise, X_{i+1} is a single element of $P \sqcup L$ not in $\bigcup_{j \le i} X_j$ incident to one element of $\bigcup_{j \le i} X_j$. Note that $X_i \subset P$ when i is even, and $X_i \subset L$ otherwise.

It models the intuitive notion of sequence of construction for the configuration (P, L): once we choose the image under ϕ of the projective base X_0, we proceed to a sequence of construction of points and lines defining at each step the image of X_{i+1} using the image of $\bigcup_{j \le i} X_j$. When X_{i+1} is formed by a single line (or point) containing only one point (or line) already constructed, the situation is underdeterminated and results either in different branches of the procedure if we search for subconfigurations, or in the introduction of a free variable if we test geometric realizability. These two situations are described separately below.

Subconfigurations — Our first task is to search for subconfigurations of a finite point – line configuration (Π, Λ) which are isomorphic to another point – line configuration (P, L). Note that it can be seen as a particular instance of subgraph isomorphism, but that the construction sequence method will significantly speed up the research.

Algorithm 1 exploits construction sequences to search for subconfigurations of a finite point – line configuration. See page 9. The progress of this algorithm depends on the choice of the initial projective base: it might create several branches for certain choices of this base, and much less for other bases. Similarly, in case of branches, the choice of the line $\ell \in L \smallsetminus D$ can also affect the number of further branches needed to complete the construction sequence. This can be easily

Algorithm 1 — Subconfigurations

```
Require: Two finite point–line configurations (P, L) and (Π, Λ).
Ensure: All subconfigurations of (Π, Λ) isomorphic to (P, L).
  Choose an arbitrary projective base (p1, p2, p3, p4) of (P, L).
  for each projective base (π1, π2, π3, π4) of (Π, Λ) do
    Initialize the images φ(pi) := πi for i ∈ [4] and the definition domain D := {p1, p2, p3, p4}.
    repeat
      Initialize a boolean NCL := false (witnessing whether we found new constructible lines).
      for each line ℓ ∈ L ∖ D do
        if there exists points p, p′ ∈ ℓ ∩ D then
          Set φ(ℓ) := φ(p) ∨ φ(p′) and update D := D ∪ {ℓ} and NCL := true.
          Check that q ∈ ℓ ⟺ φ(q) ∈ φ(ℓ) for all q ∈ P ∩ D. Otherwise reject.
        end if
      end for
      if NCL then
        Dualize P ↔ L and Π ↔ Λ and repeat.
      else
        Choose an arbitrary line ℓ ∈ L ∖ D such that there is one point p ∈ ℓ ∩ D.
        for each line λ ∈ Λ ∖ φ(D) containing φ(p) do
          Set φ(ℓ) := λ and update D := D ∪ {ℓ}.
          Check that q ∈ ℓ ⟺ φ(q) ∈ φ(ℓ) for all q ∈ P ∩ D. Otherwise reject.
          Dualize P ↔ L and Π ↔ Λ and repeat.
        end for
      end if
    until D = P ⊔ L.
    return φ.
  end for
```

optimized over all possible construction sequences. In any case, whatever choices are made, the algorithm will always end up with all subconfigurations isomorphic to (P, L) in (Π, Λ).

In our presentation of this algorithm, we are always constructing lines but we dualize at each step of the algorithm, thus inverting the role of points and lines at each step. We could have instead written twice the same code, exchanging points with lines in the second copy. We have preferred this version to shorten the presentation.

We use subconfigurations to test for example Pappus' theorem in our configurations. For that, it suffices to search in each (19_4) configuration for the non-Pappus configuration — obtained from Pappus configuration represented in Figure 1 (right) by deleting a single incidence. We illustrate examples of Pappus and non-Pappus subconfigurations of a (19_4) configuration in Figure 8. Using Algorithm 1, we could test efficiently Pappus' and Desargues' theorems in our 2 125 topological (19_4) configurations (one per duality class), and we obtain the following result.

Result 3. *Among the* 2 125 *topological* (19_4) *configurations (up to combinatorial equivalence and duality), only* 512 *configurations are compatible with both Pappus' and Desargues' theorems.*

Geometric realizability — Our second important task is to test whether a combinatorial point–line configuration (P, L) can be realized geometrically in the projective plane. Observe first that it is clearly an instance of the *Existential Theory of the Reals* (ETR): it can be expressed as a system of polynomial equalities $\mathbb{E}$ and inequalities $\mathbb{I}$ on a set Θ of real variables. A naive approach consists in assigning two variables to each point of P and to each line of L, and to construct quadratic equalities and inequalities according to the point–line incidences (one quadratic equality per incidence, and one quadratic inequality per missing incidence).

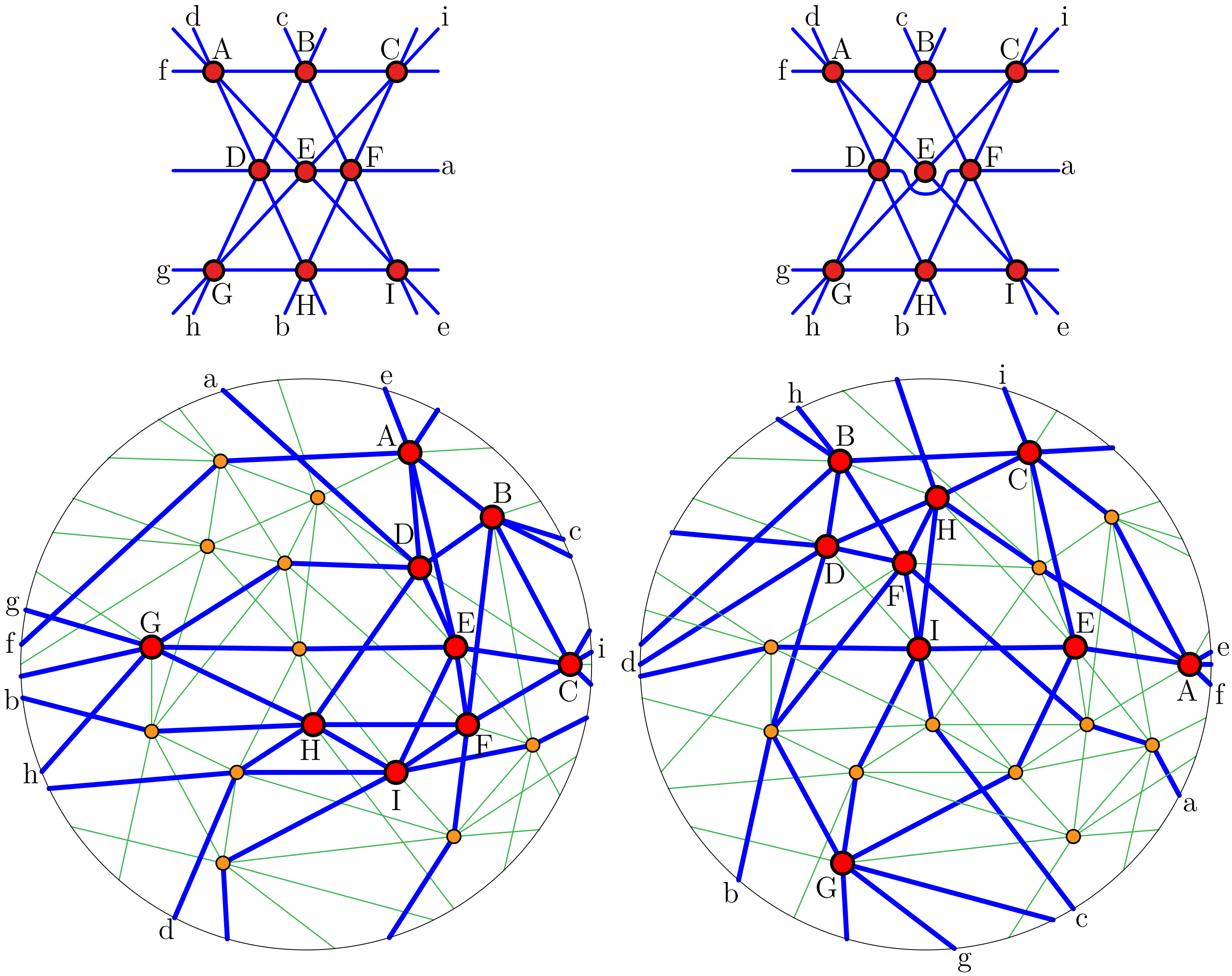

FIGURE 8. The Pappus (top left) and non-Pappus (top right) configurations as subconfigurations of a topological (19_4) configuration (bottom).

Remark 3.1. Working in the projective plane, we represent both points and lines of $\mathbb{P}$ as (pairs of antipodal) vectors. In other words, a line of $\mathbb{P}$ is represented by its normal direction. All geometric primitives on points and lines in $\mathbb{P}$ then correspond to simple computations on their representing vectors:

(i) a point is incident to a line iff their representing vectors are orthogonal, and
(ii) the vector representing the point defined by two lines is the cross product of the vectors representing these two lines. Similarly, the vector representing the line defined by two points is the cross product of the vectors representing these two points.

Algorithm 2 exploits construction sequences to test the geometric realizability of a combinatorial point – line configuration (P, L). See page 11. It still expresses the problem of geometric realizability of a configuration as an instance of ETR. However, its contribution is to reduce drastically the number of variables needed, to the price of increasing substantially the degree of the polynomials involved in the equalities and inequalitites.

Algorithm 2 — Geometric realizability

Require: A finite point – line configuration (P, L).
Ensure: Decides whether (P, L) is geometrically realizable in the projective plane $\mathbb{P}$.

Choose arbitrary projective bases (p_1, p_2, p_3, p_4) of (P, L) and $(\pi_1, \pi_2, \pi_3, \pi_4)$ of $\mathbb{P}$.
Initialize the images $\phi(p_i) := \pi_i$ for $i \in [4]$ and the definition domain $D := \{p_1, p_2, p_3, p_4\}$.
Initialize the collections of equalities $\mathbb{E} := \varnothing$, of inequalities $\mathbb{I} := \varnothing$, and of variables $\Theta := \varnothing$.
repeat
  Initialize a boolean NCL := false (witnessing whether we found new constructible lines).
  for each line $\ell \in L \smallsetminus D$ **do**
    if there exists points $p, p' \in \ell \cap D$ **then**
      Set $\phi(\ell) := \phi(p) \vee \phi(p')$ and update $D := D \cup \{\ell\}$ and NCL := true.
      for each $q \in P \cap D$ **do**
        if $q \in \ell$ **do** $\mathbb{E} := \mathbb{E} \cup \{\phi(q) \cdot \phi(\ell) = 0\}$ **else** $\mathbb{I} := \mathbb{I} \cup \{\phi(q) \cdot \phi(\ell) \neq 0\}$ **end if**.
      end for
    end if
  end for
  if not NCL **then**
    Choose an arbitrary line $\ell \in L \smallsetminus D$ such that there is one point $p \in \ell \cap D$.
    Introduce a variable $\theta \in \mathbb{R}$ and update $\Theta := \Theta \cup \{\theta\}$.
    Define a parametrization $\theta \in \mathbb{R} \mapsto \lambda_\theta$ of the lines passing through $\phi(p)$.
    Set $\phi(\ell) := \lambda_\theta$ and update $D := D \cup \{\ell\}$.
    for each $q \in P \cap D$ **do**
      if $q \in \ell$ **do** $\mathbb{E} := \mathbb{E} \cup \{\phi(q) \cdot \phi(\ell) = 0\}$ **else** $\mathbb{I} := \mathbb{I} \cup \{\phi(q) \cdot \phi(\ell) \neq 0\}$ **end if**.
    end for
  end if
  Dualize $P \leftrightarrow L$ and $\Pi \leftrightarrow \Lambda$ and repeat.
until $D = P \sqcup L$.
if the system of equalities $\mathbb{E}$ and inequalities $\mathbb{I}$ in the variables Θ has a solution $\Theta_\circ$ **then**
  Replace Θ by $\Theta_\circ$ in ϕ.
  return ϕ.
else
  Reject.
end if

Observe that in this algorithm, it is sufficient to consider only one arbitrary projective base of $\mathbb{P}$ since they are all projectively equivalent. Observe also that we have chosen again to shorten the code by dualizing the configuration at each step of the algorithm.

To solve the ETR instance $(\mathbb{E}, \mathbb{I}, \Theta)$, we use the computer algebra system MAPLE. Although the resulting system contains a priori equalities and inequalities of high degree in several variables, we observed that among the 512 topological (19_4) configurations compatible with Pappus' theorem:

(i) 10 configurations admit a complete construction sequence which never introduces any variable, but do not fulfill all required incidences. They are immediately discarded.

Example 1. The configuration given by the incidence table

lines	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s
	C	F	F	B	B	B	C	H	A	A	A	G	G	G	L	A	E	E	D
points	F	I	N	C	D	N	D	M	D	H	C	L	K	J	O	B	G	L	E
in lines	H	J	O	K	I	P	J	O	K	J	I	Q	M	N	P	E	H	M	Q
	L	K	Q	Q	O	S	M	S	S	P	R	S	P	R	R	F	I	N	R

admits the complete construction sequence

ABCD - degikp - IK - b - FJ - aj - H - q - E - s - QR - cn - GNO - fhlmor - LMPS,

which does not introduce any variable point or line. The contradiction in the construction sequence arises when we construct the lines h and o (undesired incidences R-h and H-o are forced) and when we then construct the last points L, M, P, and S (desired incidences M-m, P-o, M-r, L-r, P-m, S-i, L-o, S-l are missing, and undesired incidences S-o, M-o are forced). Although they are computed by our MAPLE code, the reader can check these additional and missing incidences by performing the given construction sequence with a dynamic geometry software like CINDERELLA.

(ii) 486 configurations admit a construction sequence which results in an ETR instance involving only one variable, and for which the equalities and inequalities of degree at most four already produce a contradiction (note that such a system can be solved by radicals). In fact, among these 496 cases, most of them already contain a contradiction in their equalities and inequalities of smaller degree. The following table shows the repartition (and percentage) of the minimal degree leading to a contradiction in the 496 construction sequences that we have considered.

degree	0	1	2	3	4
number of cases	15	92	192	132	55
proportion (%)	3	19	40	27	11

Note that each configuration could admit another construction sequence which yields a contradiction of smaller degree. We did not try to optimize further than getting, for each configuration, a construction sequence leading to a contradiction of degree at most four.

Example 2. The configuration given by the incidence table

lines	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s
points in lines	F	F	F	C	E	E	H	C	B	B	B	I	G	D	G	A	A	A	A
	H	J	L	D	H	L	J	K	D	I	C	K	N	G	I	B	C	D	N
	K	M	P	E	M	R	P	O	N	O	J	M	Q	M	J	E	G	K	O
	N	O	Q	I	Q	S	S	S	S	Q	R	R	R	P	L	F	H	L	P

admits the construction sequence

ABCD - dikpqr - E - $\boxed{\text{f}}$ - LRS - h - K - l - I - jo - GJO - bgmns - FHMNPQ - ace,

which introduces only one variable when constructing line f (boxed in the sequence). Let us follow the beginning of the construction sequence. First, we send the projective base $ABCD$ of the configuration to the projective base $\phi(A) = [1, 0, 0]$, $\phi(B) = [0, 1, 0]$, $\phi(C) = [0, 0, 1]$, and $\phi(D) = [1, 1, 1]$ of the projective plane $\mathbb{P}$. We can then construct the images of the line $\phi(d) = \phi(C \vee D) = \phi(C) \vee \phi(D) = [-1, 1, 0]$, and similarly $\phi(i) = [1, 0, -1]$, $\phi(k) = [1, 0, 0]$, $\phi(p) = [0, 0, 1]$, $\phi(q) = [0, -1, 0]$, $\phi(r) = [0, -1, 1]$. In turn, we obtain the image of the point $\phi(E) = \phi(d \wedge p) = \phi(d) \wedge \phi(p) = [1, 1, 0]$. At that stage, the construction sequence is blocked since there is no element of the configuration incident to two elements whose images are already determined. We therefore decide to construct the line $\phi(f) = [1, -1, \theta]$ containing the point $\phi(E)$ and parametrized by the variable $\theta \in \mathbb{R}$. We can then start again the construction using this last constructed line $\phi(f)$. We construct the points $\phi(L) = [\theta-1, -1, -1]$, $\phi(R) = [0, \theta, 1]$, and $\phi(S) = [1, \theta + 1, 1]$. The incidences of these points with the lines already constructed force $\theta \notin \{0, -1, 1\}$. While we keep running the construction sequence, we do not need any further variable, but we obtain many more conditions on θ (our MAPLE code produces 9 equalities and 113 inequalities). One of the inequalities simplifies to $0 \neq 0$, meaning that whatever the parameter θ is, an undesired incidence is forced. Even without considering any equality and inequality involving the variable θ, we thus conclude that the system has no solution.

(iii) the remaining 16 configurations require two variables. These last cases are a bit more complicated to handle, and we therefore start with two examples.

Example 3. The configuration given by the incidence table

lines	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s
	C	F	F	C	E	E	C	D	B	B	B	D	H	G	G	A	A	A	A
points	D	K	O	E	I	K	I	H	H	D	C	I	J	L	J	B	G	M	Q
in lines	F	L	P	J	L	P	N	M	L	J	O	K	P	N	K	E	H	N	R
	G	M	Q	M	Q	R	R	Q	R	N	S	S	S	P	O	F	I	O	S

admits the construction sequence

ABCD - ajkp - F - $\boxed{\text{b}}$ - $\boxed{\text{E}}$ - d - JM - hr - NO - cgo - GKQ - eflnqs - HILPRS - im,

which introduces only two free variables when constructing line b and point E (boxed in the sequence). As earlier, let us perform the first steps of the construction sequence. Starting from the projective base $\phi(A) = [1, 0, 0]$, $\phi(B) = [0, 1, 0]$, $\phi(C) = [0, 0, 1]$, and $\phi(D) = [1, 1, 1]$, we construct the lines $\phi(a) = [-1, 1, 0]$, $\phi(j) = [1, 0, -1]$, $\phi(k) = [1, 0, 0]$, $\phi(p) = [0, 0, 1]$, and then the point $\phi(F) = [1, 1, 0]$. We then need to introduce variables θ and ϑ in the next two steps, first for the line $\phi(b) = [1, -1, \theta]$ and then for the next point $\phi(E) = [1, \vartheta, 0]$. Observe that we immediately obtain that $\theta \neq 0$ since $C \notin b$, and that $\vartheta - 1 \neq 0$ since $E \notin a$. We can then construct the line $\phi(d) = [-\vartheta, 1, 0]$ and obtain that $\vartheta \neq 0$ since $A \notin d$. While we keep running the construction sequence, we do not need any further variables, but we obtain many conditions on θ and ϑ (our MAPLE code produces 9 equalities and 166 inequalities). Among them, the incidence between point I and line q leads to the equation $\theta^2\vartheta(\vartheta-1)^3 = 0$, which is already impossible since $\theta \neq 0$, $\vartheta - 1 \neq 0$, and $\vartheta \neq 0$.

Example 4. The configuration given by the incidence table

lines	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s
	D	F	F	C	E	E	C	C	B	B	B	G	D	J	L	A	A	A	A
points	F	H	I	E	G	I	D	H	J	G	C	H	K	K	O	B	D	L	O
in lines	G	K	N	J	K	Q	M	P	M	N	I	Q	P	N	R	E	H	M	P
	J	L	O	L	M	R	O	R	R	P	S	S	S	Q	S	F	I	N	Q

admits the construction sequence

ABCD - gkpq - I - $\boxed{\text{a}}$ - F - c - O - s - $\boxed{\text{E}}$ - df - JQ - in - MNR - ehjor - GHKLPS - blm,

which introduces only two free variables when constructing line a and point E (boxed in the sequence). As earlier, let us perform the first steps of the construction sequence. Starting from the projective base $\phi(A) = [1, 0, 0]$, $\phi(B) = [0, 1, 0]$, $\phi(C) = [0, 0, 1]$, $\phi(D) = [1, 1, 1]$, we construct the lines $\phi(g) = [-1, 1, 0]$, $\phi(k) = [1, 0, 0]$, $\phi(p) = [0, 0, 1]$, $\phi(q) = [0, -1, 1]$, and the point $\phi(I) = [0, -1, -1]$. We then need to introduce a free variable θ for the line $\phi(a) = [1, -\theta - 1, \theta]$. From this line, we construct the point $\phi(F) = [-1 - \theta, -1, 0]$, the line $\phi(c) = [1, -1 - \theta, 1 + \theta]$, the point $\phi(O) = [-1 - \theta, -1 - \theta, -\theta]$ and then the line $\phi(s) = [0, \theta, -1 - \theta]$. Again, we introduce a new variable ϑ for the point $\phi(E) = [1, \vartheta, 0]$. From this point, we construct the lines $\phi(d) = [-\vartheta, 1, 0]$ and $\phi(f) = [-\vartheta, 1, -1]$, then the points $\phi(J) = [-\theta, -\theta\vartheta, 1 - \vartheta - \theta\vartheta]$ and $\phi(Q) = [-1, -\theta\vartheta - \vartheta, -\theta\vartheta]$, and so on until we finally construct the line $\phi(m)$. Along the remaining construction sequence, we do not need any further variable, but we obtain many conditions on θ and ϑ (our MAPLE code produces 8 equalities and 149 inequalities). Among them, we obtain $\theta\vartheta + \vartheta - 1 \neq 0$ since $J \notin f$, while $(\theta\vartheta + \vartheta - 1)\big((2\theta^2 + 3\theta + 1)\vartheta^2 - (2\theta^2 + 3\theta + 1)\vartheta + \theta\big) = 0$ since $L \in r$ and $(\theta\vartheta + \vartheta - 1)\big((3\theta^3 + 3\theta^2 - \theta - 1)\vartheta^2 - (2\theta^3 + 5\theta^2 - 2)\vartheta + (\theta^2 + \theta - 1)\big) = 0$ since $G \in j$. We therefore obtain two equations of degree 2 in ϑ, from which we can eliminate

$$\vartheta = \frac{\theta^3 - 3\theta^2 + 1}{(2\theta + 1)(\theta^3 - 2\theta^2 - \theta + 3)}.$$

Plugging in this value of ϑ, we obtain a system of polynomial equalities and inequalities involving only one variable θ. We can again solve the subsystem of equations of degree at most 4 in θ and check that none of the resulting solutions yields a solution for the initial system, which shows that this configuration is not geometrically realizable.

Among the 16 configurations involving two variables,

- 12 cases can be handled as in Example 3. Namely, at least one equality of $\mathbb{E}$ factors into smaller polynomials which all appear as factors of at least one inequality of $\mathbb{I}$, thus providing a simple contradiction, although the construction sequence involves two variables.
- the remaining 4 cases can be handled as in Example 4. Namely, after simplification by all factors of the inequalities of $\mathbb{I}$, there are always two equalities of $\mathbb{E}$ of degree two in one of the variables. We can therefore eliminate this variable to obtain a simplified system of equalities and inequalities involving a single variable. This system can be proved to have no solution by considering only equalities of degree at most 4 and proving that the resulting solutions do not yield solutions of the initial system.

In particular, even for the cases involving two variables, we did not need any sophisticated techniques (for example based on Gröbner bases) to ensure non feasibility of all these instances of ETR. All computations were handled with the computer algebra system MAPLE.

This concludes our study of geometric (19_4) configurations, and leads to the following surprising statement, which closes one of the last remaining cases in the quest for (n_4) configurations.

Result 4. *There is no geometric* (19_4) *configuration.*

References

[BGS09] Jürgen Bokowski, Branko Grünbaum, and Lars Schewe. Topological configurations (n_4) exist for all $n \geq 17$. *European J. Combin.*, 30(8):1778–1785, 2009.

[BP13] Jürgen Bokowski and Vincent Pilaud. Enumerating topological (n_k)-configurations. To appear in *Comput. Geom.*, 2013.

[BS05] Jürgen Bokowski and Lars Schewe. There are no realizable 15_4- and 16_4-configurations. *Rev. Roumaine Math. Pures Appl.*, 50(5-6):483–493, 2005.

[BS13] Jürgen Bokowski and Lars Schewe. On the finite set of missing geometric configurations (n_4). *Comput. Geom.*, 46(5):532–540, 2013.

[Grü00] Branko Grünbaum. Connected (n_4) configurations exist for almost all n. *Geombinatorics*, 10(1):24–29, 2000.

[Grü02] Branko Grünbaum. Connected (n_4) configurations exist for almost all n—an update. *Geombinatorics*, 12(1):15–23, 2002.

[Grü06] Branko Grünbaum. Connected (n_4) configurations exist for almost all n—second update. *Geombinatorics*, 16(2):254–261, 2006.

[Grü09] Branko Grünbaum. *Configurations of points and lines*, volume 103 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2009.

[OC12] Octavio Páez Osuna and Rodolfo San Agustín Chi. The combinatorial (19_4) configurations. *Ars Math. Contemp.*, 5(2):231–237, 2012.

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